Creep Analysis of Circular Cylindrical Shells

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Governing equations for nonlinear creep analysis of thin circular cylindrical shells are derived based upon a power law for creep. These equations are solved for a clamped shell subjected to uniform internal pressure. The method of solution is one of iteration where the starting solution is that for the linear case. Stress resultants and deflections are presented for various values of the parameters involved.

1. Introduction

THE creep analysis of thin circular cylindrical shells, subjected to an axially symmetric load, has been the subject of several papers in recent years. Onat and Yuksel[1] obtained some solutions for sandwich shells using a power law in conjunction with the Tresca criterion of plasticity theory. Bieniek and Freudenthal[2] obtained approximate solutions for some specific shell problems with the use of strain energy methods. In this analysis, however, the creep relations used did not include terms that couple effects in the axial and circumferential directions. Gemma[3, 4] employed various simplifications to treat similar problems. Most recently, Calladine[5] and Rozenblum[6] have obtained some approximate solutions by a different approach, which is based on the concept of nesting surfaces established by Calladine and Drucker.[7] In all of these investigations, various simplifications have been made in the development of equations governing the specific problems considered. The present report is concerned with the creep analysis of thin circular shells under axially symmetric lateral pressure. It is based on a creep power law and the Mises criterion. The analysis neglects beam column effects and assumes that the small deflection theory of shells is valid.

Whereas Sec. 2 develops the governing equations, Sec. 3 presents complete solutions by an iterative technique for the case of a clamped shell under uniform internal pressure. Stresses and deflections are obtained for various values of the governing parameters. The paper ends with some concluding remarks.

2. Governing Equations

The present analysis is based on the creep law

$$(\epsilon_{ij})_{\text{creep}} = CJ_2{}^m s_{ij}{}^{1/q} \tag{2.1}$$

Now, Hoff[8] has shown that

$$\epsilon_{ij} = CJ_2{}^m s_{ij} \tag{2.2}$$

is the elastic analog of Eq. (2.1). In these equations, ϵ_{ij} is the strain tensor, $s_{ij} = \sigma_{ij} - (\frac{1}{3})\sigma_{kk}\,\delta_{ij}$ is the stress deviation tensor, $J_2 = \frac{1}{2}s_{ij}s_{ij}$ is the second invariant of the tensor s_{ij}, and C, m, and q are temperature-dependent material constants. It also can be shown[9] that the use of Eq. (2.2) is formally identical to the use of the Mises flow rule of plasticity theory. In the present work, the analysis is based on Eq. (2.2).

A thin circular cylindrical shell of mean radius R and length L under axially symmetric load is considered (Fig. 1). The load consists of an axially symmetric lateral pressure $P(X)$, a constant end stress resultant N_0, constant axial end moments M_{x1} and M_{x2} per unit length at the ends 1 and 2, respectively, and constant radial end shears Q_1 and Q_2. In accordance with thin-shell theory, the stresses σ_{xz}, $\sigma_{\theta z}$, and σ_{zz} are set equal to zero, where X and θ are axial and circumferential coordinates in the median plane of the shell, and Z is the radial coordinate (Fig. 1). Since the problem considered has axial symmetry, $\sigma_{\theta x} = 0$. Hence, the only stresses present are σ_x and σ_θ.

In order to simplify the analysis, it is assumed that the wall of the shell is made of an ideal sandwich section. This sandwich section is composed of two thin sheets, each of thickness $h/2$ and separated by a core of depth b-h (Fig. 2). It is assumed that the core is infinitely strong in shear but carries no membrane stresses. The stress resultants for this problem are also indicated in Fig. 2. From equilibrium considerations it follows that

$$\begin{aligned} \sigma_{xe} &= N_x/h - 2M_x/bh \qquad & \sigma_{xi} &= N_x/h + 2M_x/bh \\ \sigma_{\theta e} &= N_\theta/h - 2M/bh \qquad & \sigma_{\theta i} &= N_\theta/h + 2M_\theta/bh \end{aligned} \tag{2.3}$$

where subscripts e and i refer to the external and internal sheets, respectively. Thus, Eqs. (2.3) together with Eq. (2.2) yield

$$\left.\begin{aligned} \epsilon_{xe} &= C(G_e/3b^2h^2)^m[2(bN_x - 2M_x) - (bN_\theta - 2M_\theta)]/3bh \\ \epsilon_{xi} &= C(G_i/3b^2h^2)^m[2(bN_x + 2M_x) - (bN_\theta + 2M_\theta)]/3bh \\ \epsilon_{\theta e} &= C(G_e/3b^2h^2)^m[2(bN_\theta - 2M_\theta) - (bN_x - 2M_x)]/3bh \\ \epsilon_{\theta i} &= C(G_i/3b^2h^2)^m[2(bN_\theta + 2M_\theta) - (bN_x + 2M_x)]/3bh \end{aligned}\right\} \tag{2.4}$$

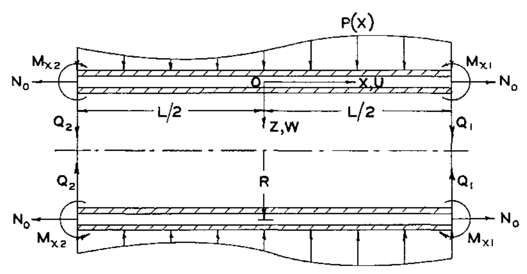

Fig. 1 Shell geometry and load.

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In these equations,

$$
\begin{aligned}
G_e &= (bN_x - 2M_x)^2 - (bN_x - 2M_x)(bN_\theta - 2M_\theta) + (bN_\theta - 2M_\theta)^2 \\
G_i &= (bN_x + 2M_x)^2 - (bN_x + 2M_x)(bN_\theta + 2M_\theta) + (bN_\theta + 2M_\theta)^2
\end{aligned}
\tag{2.5}
$$

Furthermore, assuming that small-deflection theory is valid, the strains for this problem may be expressed as follows (Ref. 10, p. 212):

$$
\epsilon_{xe} = \frac{dU}{dX} + \frac{b}{2}\frac{d^2W}{dX^2} \qquad \epsilon_{xi} = \frac{dU}{dX} - \frac{b}{2}\frac{d^2W}{dX^2}
$$
$$
\epsilon_{\theta e} = \epsilon_{\theta i} = -W/R \tag{2.6}
$$

where U and W are displacements in the axial and radial directions, respectively (Fig. 1). Hence, the compatibility conditions are

$$
\epsilon_{xe} - \epsilon_{xi} + Rb(d^2\epsilon_{\theta e}/dX^2) = 0 \qquad \epsilon_{\theta e} - \epsilon_{\theta i} = 0 \tag{2.7}
$$

Finally, the stress resultants must satisfy the equilibrium equations (Ref. 10, p. 209),

$$
N_x = N_0 \qquad N_\theta = -R[P + (d^2M_x/dX^2)]
$$
$$
Q = dM_x/dx \tag{2.8}
$$

Equations (2.4, 2.7, and 2.8) constitute nine equations involving nine unknowns.

In order to solve the foregoing set of equations, it is convenient to define the following dimensionless quantities:

$$
\left.
\begin{aligned}
x &= \frac{2X}{L} & \beta &= \left(\frac{3^{1/2}L^2}{8Rb}\right)^{1/2} & p &= \frac{P}{P^*} \\
q &= \frac{3^{1/2}QL}{2RbP^*} & n_x &= \frac{N_x}{RP^*} & n_\theta &= \frac{N_\theta}{RP^*} \\
m_x &= \frac{3^{1/2}M_x}{RbP^*} & m_\theta &= \frac{3^{1/2}M_\theta}{RbP^*} & g &= \frac{G}{RbP^*} \\
w &= \frac{3^{m+1}}{2RC}\left(\frac{h}{RP^*}\right)^{2m+1}W & u &= \frac{3^{m+1}}{LC}\left(\frac{h}{RP^*}\right)^{2m+1}U
\end{aligned}
\right\}
\tag{2.9}
$$

In these equations, P^* is a suitable reference pressure. With the use of these definitions, substitution of Eqs. (2.4) into Eqs. (2.7) yields

$$
[(\tfrac{3}{4})^{1/2}(2n_\theta - n_x) - (2m_\theta - m_x)]g_e^m - [(\tfrac{3}{4})^{1/2}(2n_\theta - n_x) + (2m_\theta - m_x)]g_i^m = 0 \tag{2.10a}
$$

$$
[(\tfrac{3}{4})^{1/2}(2n_x - n_\theta) - (2m_x - m_\theta)]g_e^m - [(\tfrac{3}{4})^{1/2}(2n_x - n_\theta) + (2m_x - m_\theta)]g_i^m + (3^{1/2}/2\beta^2)\{[(\tfrac{3}{4})^{1/2}(2n_\theta - n_x) - (2m_\theta - m_x)]g_e^m\}'' = 0 \tag{2.10b}
$$

where primes indicate differentiations with respect to x, and from Eq. (2.5),

$$
\left.
\begin{aligned}
g_e &= [(n_x^2 - n_xn_\theta + n_\theta^2) + (\tfrac{4}{3})(m_x^2 - m_xm_\theta + m_\theta^2)] - (12)^{1/2}[n_x(2m_x - m_\theta) + n_\theta(2m_\theta - m_x)] \\
g_i &= [(n_x^2 - n_xn_\theta + n_\theta^2) + (\tfrac{4}{3})(m_x^2 - m_xm_\theta + m_\theta^2)] + (12)^{1/2}[n_x(2m_x - m_\theta) + n_\theta(2m_\theta - m_x)]
\end{aligned}
\right\}
\tag{2.11}
$$

Finally, the equilibrium equations (2.8) assume the dimensionless forms,

$$
n_x = N_0/RP^* = n_0 \tag{2.12a}
$$
$$
n_\theta = -[p + (m_x''/2\beta^2)] \tag{2.12b}
$$
$$
q = m_x' \tag{2.12c}
$$

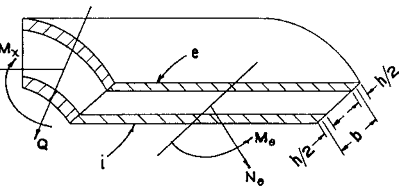

Fig. 2 Stress resultants in sandwich wall.

As a preliminary to the solution of the governing equations, Eq. (2.10a) is rewritten in the form,

$$
m_\theta = \frac{m_x}{2} + \left(\frac{3^{1/2}}{4}\right)\frac{(g_e^m - g_i^m)(2n_\theta - n_0)}{(g_e^m + g_i^m)} \tag{2.13}
$$

with the aid of Eq. (2.12a). Next, Eq. (2.10b) is simplified with the use of Eqs. (2.12a, 2.12b, and 2.13) to read

$$
m_x^{iv} + 4\beta^4m_x = 2\beta^2F \tag{2.14a}
$$

where

$$
F = -p'' + m[g_eg_e'' + (m-1)(g_e')^2]\frac{(2n_\theta - n_0)g_e^{m-2}}{g_e^m + g_i^m} + \frac{2mg_e'}{g_e}\left[\frac{(2n_\theta - n_0)g_e^m}{g_e^m + g_i^m}\right]' - \left[\frac{(2n_\theta - n_0)(g_e^m - g_i^m)}{2(g_e^m + g_i^m)}\right]'' + \beta^2\left(m_x + \frac{3^{1/2}n_0}{2}\right)\frac{g_e^m - g_i^m}{g_e^m} \tag{2.14b}
$$

Since the stress resultant n_θ is a function of m_x, Eqs. (2.12b, 2.13, and 2.14) are simultaneous equations in the unknowns m_x and m_θ. With the solution of these equations, the nondimensional displacements may be obtained from Eqs. (2.4, 2.6, and 2.9) as

$$
\left.
\begin{aligned}
u &= \frac{1}{4}\int_0^x \left[(2n_0 - n_\theta)(g_e^m + g_i^m) - \left(\frac{4}{3}\right)^{1/2}(2m_x - m_\theta)(g_e^m - g_i^m)\right]dx \\
w &= [(2m_\theta - m_x)/3^{1/2} - (2n_\theta - n_0)/2]g_e^m
\end{aligned}
\right\}
\tag{2.15}
$$

In the first of the foregoing equations, the axial displacement at midspan has been set equal to zero. The procedure for the solution of Eqs. (2.13) and (2.14) will be illustrated in the next section in obtaining results for a particular example.

3. Clamped Shell Under Uniform Internal Pressure

The example considered is that of a shell subjected to a uniform internal pressure $P = -P_0$. The shell is assumed to be clamped at both ends in supports that are free to move in the axial direction. Hence, choosing $P^* = P_0$, it is seen that $n_x = n_0 = 0$ and $p = -1$. Furthermore, conditions of symmetry show that $m_{x1} = m_{x2} = m_0$ and $q_1 = q_2 = q_0$. It then follows from Eqs. (2.11, 2.13, and 2.14b) that

$$
g_e = g_i = g = m_x^2 + n_\theta^2 \tag{3.1a}
$$
$$
m_\theta = m_x/2 \tag{3.1b}
$$
$$
F = [m(g''n_\theta + 2g'n_\theta')]/g + [m(m-1)(g')^2n_\theta]/g^2 \tag{3.1c}
$$

Equation (3.1a) together with Eq. (2.12b) shows that

$$
g'' = [-(m_x^{iv} + 4\beta^4m_x)n_\theta/\beta^2] + 2(m_x'^2 + 2\beta^2m_x + n_\theta'^2) \tag{3.2}
$$

Hence, substitution of Eqs. (3.1a, 3.1c, and 3.2) into Eqs. (2.14) and simplification yield

$$
m_x^{iv} + 4\beta^4m_x = 2\beta^2f \tag{3.3a}
$$

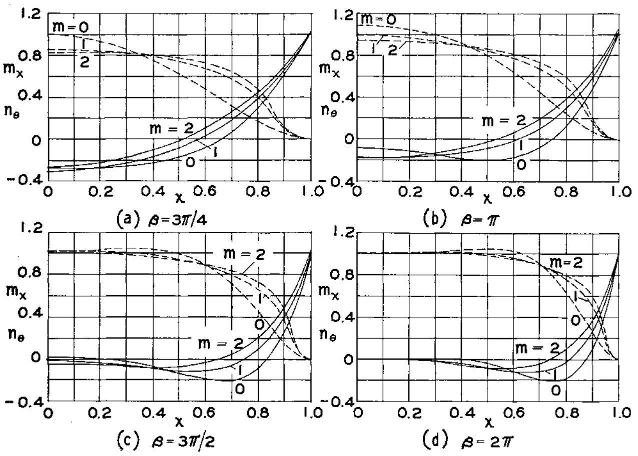

Fig. 3 Stress resultants.

where

$$= \frac{2m[2(m_x m_x' + n_\theta n_\theta')n_\theta' + (m_x'^2 + 2\beta^2 m_x + n_\theta'^2)n_\theta]}{m_x^2 + (2m+1)n_\theta^2} + \frac{4m(m-1)(m_x m_x' + n_\theta n_\theta')^2 n_\theta}{[m_x^2 + (2m+1)n_\theta^2](m_x^2 + n_\theta^2)} \quad (3.3b)$$

A direct solution of Eq. (3.3a) is extremely difficult. However, for $m = 0$ the equation can be readily solved, since the right-hand side of the equation is zero. This fact suggests an iterative method of solution for any arbitrary value of m. Thus, Eq. (3.3a) is first solved for m_x with $f = 0$. This solution, in conjunction with (2.12b) is used to evaluate f as a function of x for a given value of m. The evaluated f is then substituted in Eq. (3.3a) to give a second solution of m_x, which in turn yields a second value of f. The procedure is repeated until two successive solutions are sufficiently close to be acceptable.

In accordance with the procedure outlined, the differential equation for the jth iteration turns out to be

$$m_{x,\,j+1}^{iv} + 4\beta^4 m_{x,\,j+1} = 2\beta^2 f_j(m_{x,\,j}, m_{x,\,j}', n_{\theta,\,j}, n_{\theta,\,j}') \qquad j = 0, 1, 2, \ldots N \quad (3.4)$$

where $j = 0$ corresponds to the first approximation $m = 0$, and N is the number of iterations deemed sufficient for the accuracy of m_x. The general solution to this nonhomogeneous linear differential equation follows as (Ref. 11, Secs. 6.11 and 6.21)

$$m_{x,\,j+1} = A_j \sin\beta x \sinh\beta x + B_j \cos\beta x \cosh\beta x - \frac{1}{2\beta}\int_0^x [\cos\beta(x-t)\sinh\beta(x-t) - \sin\beta(x-t)\cosh\beta(x-t)]f_j(t)dt \quad (3.5)$$

Since m_x is symmetric about midspan ($x = 0$), only two constants of integration, A_j and B_j, appear in the solution. The evaluation of these constants will be discussed subsequently.

At this point, the following notation is introduced for convenience:

$$\psi(x) = \cos\beta x \cosh\beta x \qquad \xi(x) = \cos\beta x \sinh\beta x$$
$$\eta(x) = \sin\beta x \cosh\beta x \qquad \rho(x) = \sin\beta x \sinh\beta x \quad (3.6a)$$

Table 1 End moments and shears

β \ m	m_0 0	m_0 1	m_0 2	q_0 0	q_0 1	q_0 2
$3\pi/4$	1.037	1.021	1.03	4.80	4.22	4.0
π	0.100	1.015	1.05	6.26	5.38	5.1
$3\pi/2$	0.100	0.999	1.04	9.43	8.04	7.5
2π	0.100	1.001	1.03	12.57	10.75	10.0

$$H_{1j}(x) = \frac{1}{2\beta}\int_0^x \psi(t)f_j(t)dt \qquad H_{2j}(x) = \frac{1}{2\beta}\int_0^x \xi(t)f_j(t)dt \quad (3.6b)$$

$$H_{3j}(x) = \frac{1}{2\beta}\int_0^x \eta(t)f_j(t)dt \qquad H_{4j}(x) = \frac{1}{2\beta}\int_0^x \rho(t)f_j(t)dt$$

$$\left.\begin{aligned} I_{1j}(x) &= \xi H_{1j} - \psi H_{2j} + \rho H_{3j} - \eta H_{4j} \\ I_{2j}(x) &= \eta H_{1j} - \rho H_{2j} - \psi H_{3j} + \xi H_{4j} \\ I_{3j}(x) &= \rho H_{1j} - \eta H_{2j} - \xi H_{3j} + \psi H_{4j} \\ I_{4j}(x) &= \psi H_{1j} - \xi H_{2j} + \eta H_{3j} - \rho H_{4j} \end{aligned}\right\} \quad (3.6c)$$

With the use of this notation, Eqs. (3.5) and (2.12b) yield the following stress resultants and their derivatives:

$$m_{x,\,j+1} = A_j\rho + B_j\psi - I_{1j} + I_{2j} \quad (3.7a)$$
$$m_{x,\,j+1}' = \beta[(A_j + B_j)\xi + (A_j - B_j)\eta + 2I_{3j}] \quad (3.7b)$$
$$n_{\theta,\,j+1} = 1 - A_j\psi + B_j\rho - I_{1j} - I_{2j} \quad (3.7c)$$
$$n_{\theta,\,j+1}' = \beta[(A_j + B_j)\eta - (A_j - B_j)\xi - 2I_{4j}] \quad (3.7d)$$

If the constants A_j and B_j are known, successive values of f_j [Eq. (3.3b)] can be evaluated by means of the preceding equations.

Since the boundary conditions of the problem are on the displacements, the displacement solutions must be considered for the evaluation of the constants A_j and B_j. To this end, it is observed that, in the present case, Eqs. (2.15) reduce to

$$u = \frac{1}{2}\int_0^x w dx \quad (3.8a)$$

$$w = -n_\theta(m_x^2 + n_\theta^2)^m \quad (3.8b)$$

Since the ends of the shell are clamped, the boundary conditions are $w(1) = w'(1) = 0$. Hence, it follows from Eq. (3.8b) that $n_\theta(1) = n_\theta'(1) = 0$. Although these conditions may be used to determine the constants A_j and B_j, they give rise to successive values of m_x and n_θ which do not converge. This difficulty is overcome by adopting an alternative procedure that entails enforcing a boundary condition on m_x. Thus, choosing some value of m_0 and satisfying the boundary conditions

$$n_\theta(1) = 0 \qquad m_x(1) = m_0 \quad (3.9)$$

A_j and B_j are evaluated from Eqs. (3.7) as

$$A_j = \frac{[\rho(1) - \psi(1)]I_{1j}(1) - [\rho(1) + \psi(1)]I_{2j}(1) + m_0\rho(1) + \psi(1)}{[\rho(1)]^2 + [\psi(1)]^2} \quad (3.10)$$

$$B_j = \frac{[\rho(1) + \psi(1)]I_{1j}(1) + [\rho(1) - \psi(1)]I_{2j}(1) + m_0\psi(1) - \rho(1)}{[\rho(1)]^2 + [\psi(1)]^2}$$

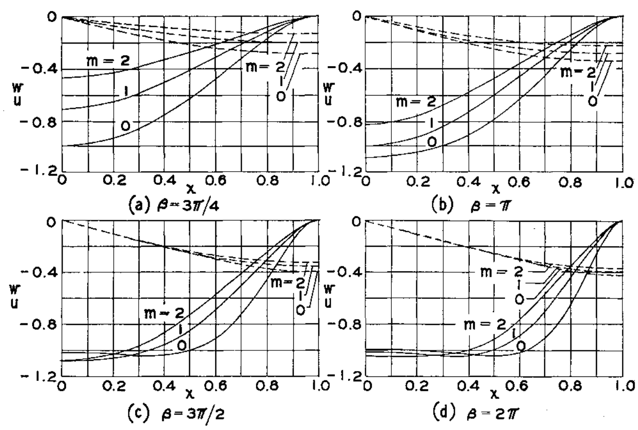

Fig. 4 Deflections.

With these constants, Eqs. (3.7) are then used to determine successive values of m_x and n_θ until they converge. The corresponding value of $n_\theta'(1)$ is determined from Eq. (3.7d) to insure that it vanishes. If it does not, another value of m_0 is chosen, and the procedure is repeated until the condition $n_\theta'(1) = 0$ is satisfied. These calculations were performed on an IBM 7090 computer.

The preceding computations were carried out for values of $m = 0, 1,$ and 2 and for values of $\beta = 3\pi/4, \pi, 3\pi/2,$ and 2π. The plots of m_x and n_θ vs x, and u and w vs x are shown plotted in Figs. 3 and 4, respectively. Furthermore, m_x vs $\beta(1 - x)/\pi$ for constant values of m is shown plotted in Fig. 5. The latter plot is useful in showing the effects of variations in β. Finally, the end moments m_0 and the end shears q_0 for the various values of the parameters are given in Table 1.

4. Concluding Remarks

The following conclusions can be drawn from the results presented in this paper. From the table in the preceding section, it appears that, for the range of β considered, the end moments m_0 are relatively insensitive to variations in the creep exponent m. Furthermore, Fig. 3 shows that, as m increases, the moment m_x decays less rapidly with distance from the end of the shell. In a similar manner, it is seen that the stress resultant n_θ decays less rapidly away from the center as m increases. Next, Fig. 5 shows that, for all values of the exponent m and length parameter $\beta = 2\pi$, the moment m_x is negligibly small over a significant interval at the center of the shell.

It appears reasonable to compare the present stress solutions with those obtained by Bienick and Freudenthal for long clamped circular cylindrical shells in Ref. 2. The essential purpose of their investigation was to study the relative rapidity of decay of the moment m_x from its peak intensity at the clamped edge of the shell. In their analysis, the creep relations were simplified by ignoring terms that couple effects in the axial and circumferential directions, and the resulting equations were solved approximately by an extremum principle. In the use of this principle, however, the range of integration for evaluating the work done was split into two parts: one over which the bending moments were significant and the other over which the work done by the bending moments could be neglected. Thus, the parameter defining the transition length was then determined. The stress solutions of this analysis over a length $\beta = 2\pi$ from the clamped edge are reproduced in Fig. 6. The figure also shows the present solution for the same shell length. Although the two n_θ distributions agree reasonably well for all values of the exponent m, the moment distributions differ significantly for $m \neq 0$. This discrepancy appears to be attributable to the approximations used in Ref. 2.

Finally, it can be seen from Fig. 4 that the maximum radial deflection for $m = 0$ does not occur at the center, if the shell is sufficiently long (β large). However, as the index m increases, this maximum approaches the center.

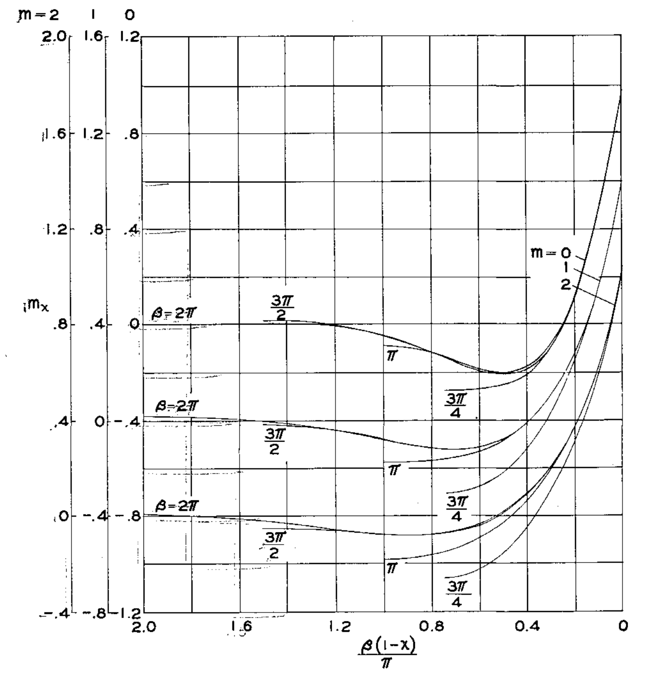

Fig. 5 Axial moment.

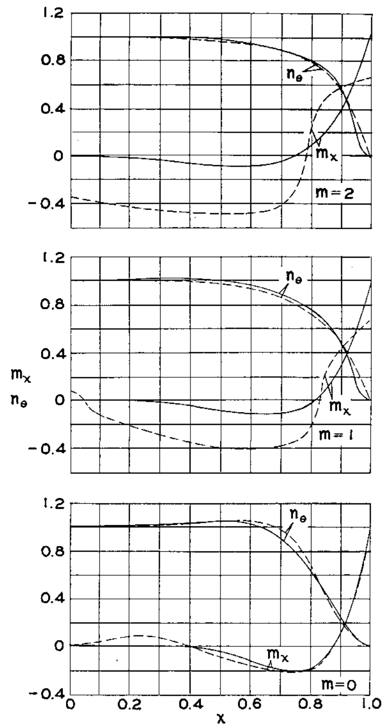

Fig. 6 Comparison of present stress resultant solutions with those of Ref. 2.

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